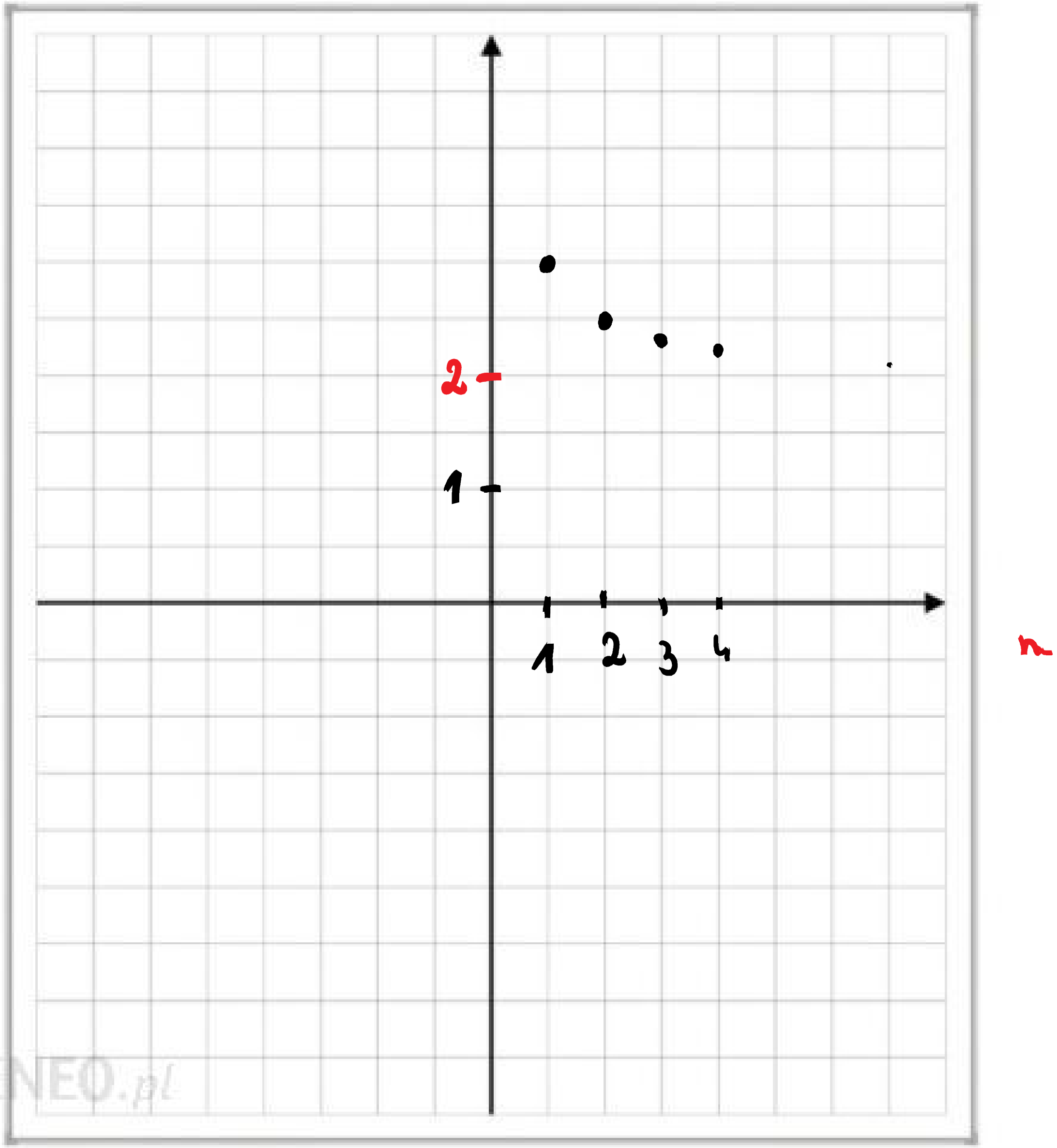


$$a_n = 2 + \frac{1}{n}$$

$$\lim_{n \to \infty}\left(2 + \frac{1}{n}\right) = 2$$

$$a_1 = 2 + \frac{1}{1} = 3$$

$$a_2 = 2 + \frac{1}{2} = 2\frac{1}{2}$$

$$a_3 = 2\frac{1}{3}$$

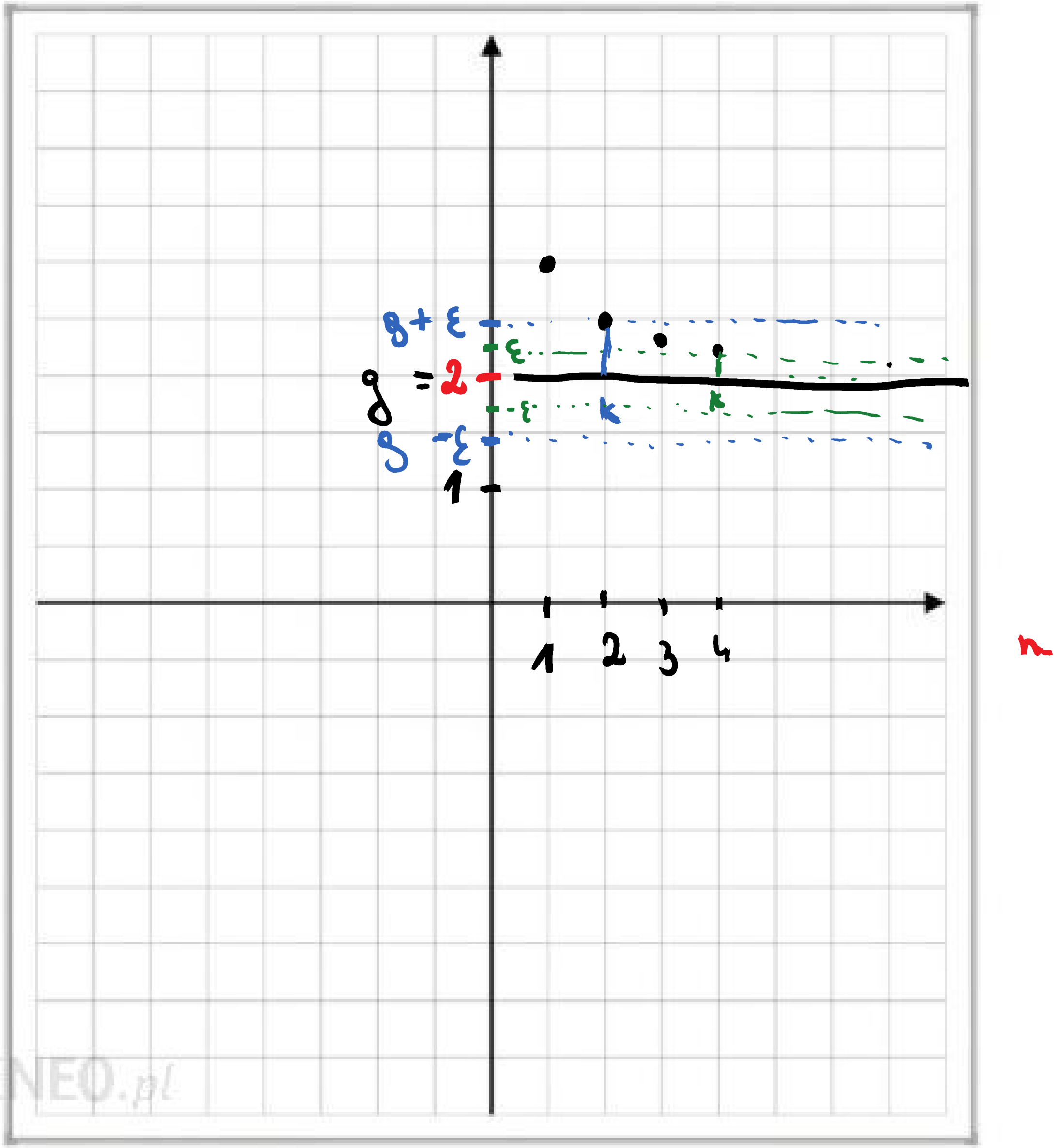
$$a_4 = 2\frac{1}{4}$$

$$\vdots$$

$$a_{100} = 2\frac{1}{100}$$

$\lim\limits_{n \to \infty} a_n$ – granica ciągu przy n zmierzającym do nieskończoności

„limes"



$$\lim_{n \to \infty} a_n =$$

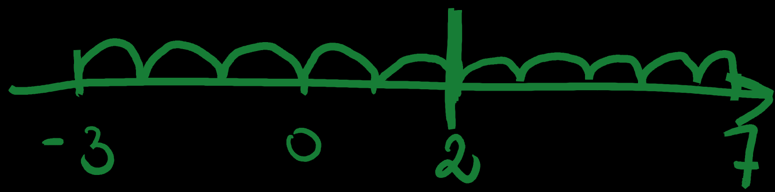
Ciąg (an) ma granicę równą g, jeśli dla każdego ε>0 istnieje liczba naturalna k taka, że dla wszystkich n>k zachodzi nierówność: |an – g|<ε Jeśli ciąg (an) ma granicę równą g, to mówimy, że jest zbieżny do g.

$\varepsilon > 0$ g - granica

$k \in \mathbb{N}$ $n > k$

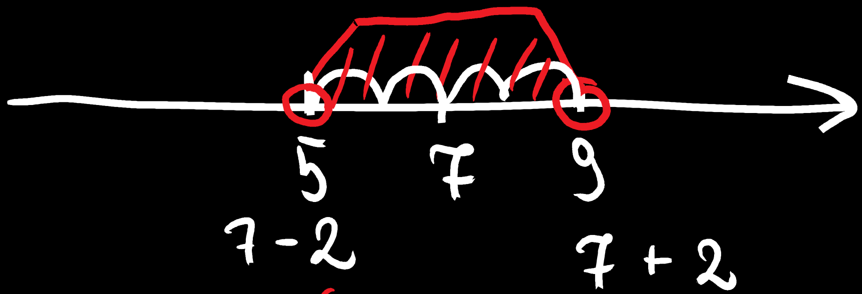
$$|a_n - g| < \varepsilon$$

$$|x - 2| = 5$$



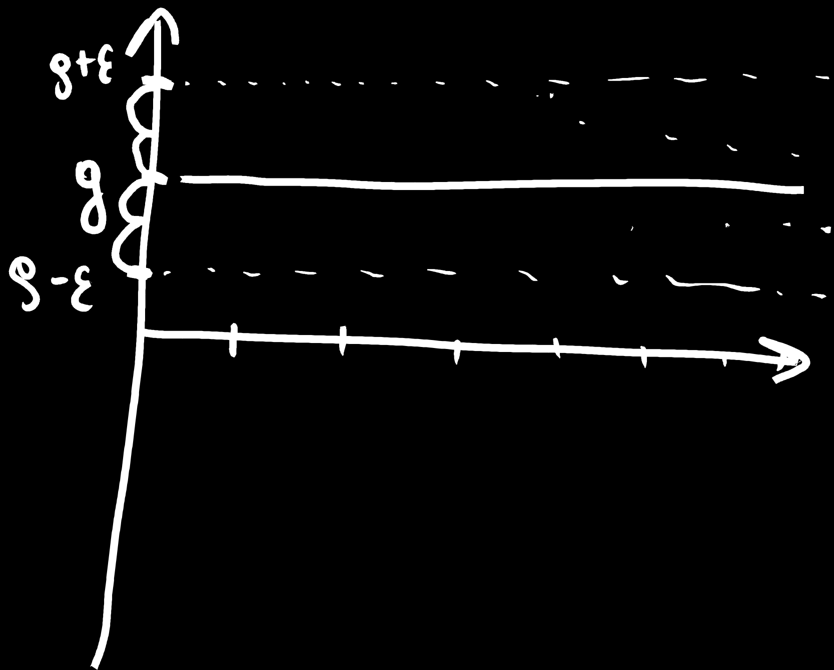
$x = -3$ $x = 7$

$|x-7| < 2$



$x \in (5, 9)$

$|a_n - g| < \varepsilon$



$$a_n = \frac{4}{n}$$

$$\varepsilon = \frac{1}{10}$$

a) $\varepsilon = \frac{1}{10}$

$a_1 = 4$

$a_2 = 2$

$a_3 = \frac{4}{3}$

$a_4 = 1$

$a_5 = \frac{4}{5}$

$\vdots$

$a_{100} = \frac{1}{25}$

$a_{1000} = \frac{1}{250}$

$|a_n - g| < \varepsilon$ $\quad n \in \mathbb{N}$

$|a_n - 0| < \varepsilon$

$g = 0$

$$\left|\frac{4}{n} - 0\right| < \frac{1}{10}$$

$$\left|\frac{4}{n}\right| < \frac{1}{10}$$

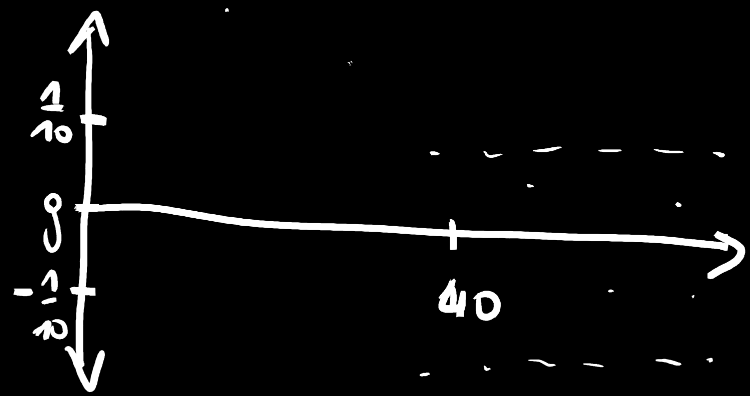
$$\frac{4}{n} < \frac{1}{10} \quad / \cdot n > 0$$

$$4 < \frac{1}{10} n \quad / \cdot 10$$

$$40 < n$$

$$n > 40$$

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$$



$\varepsilon = \frac{1}{50}$

$|a_n - 0| < \varepsilon \qquad a_n = \frac{1}{n}$

$\left|\frac{1}{n} - 0\right| < \frac{1}{50}$

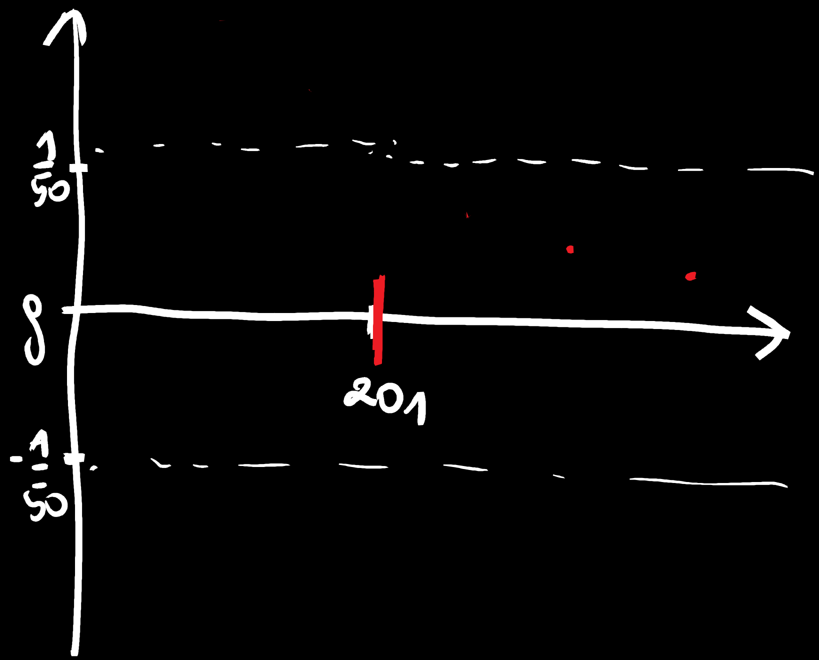
$\left|\frac{1}{n}\right| < \frac{1}{50}$

$\frac{1}{n} < \frac{1}{50} \quad / \cdot n$

$1 < \frac{1}{50} n \quad / \cdot 50$

$200 < n$

$n > 200$



3.

$a_n = \frac{n+1}{n} \qquad g = 1 \qquad \varepsilon = \frac{1}{50} \qquad n \in \mathbb{N}$

$|a_n - g| < \varepsilon$

$\left|\frac{n+1}{n} - 1\right| < \frac{1}{50}$

$\left|\frac{n}{n} + \frac{1}{n} - 1\right| < \frac{1}{50}$

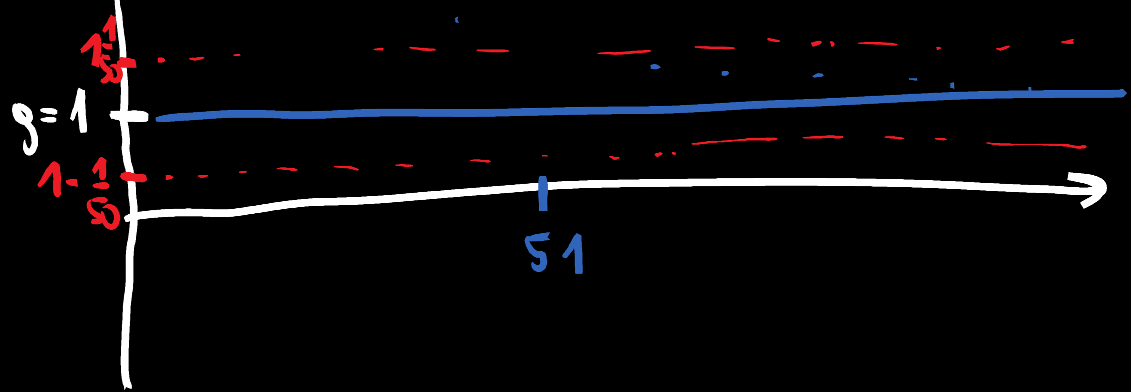
$\left|1 + \frac{1}{n} - 1\right| < \frac{1}{50}$

$\left|\frac{1}{n}\right| < \frac{1}{50}$

$\frac{1}{n} < \frac{1}{50} \quad / \cdot n$

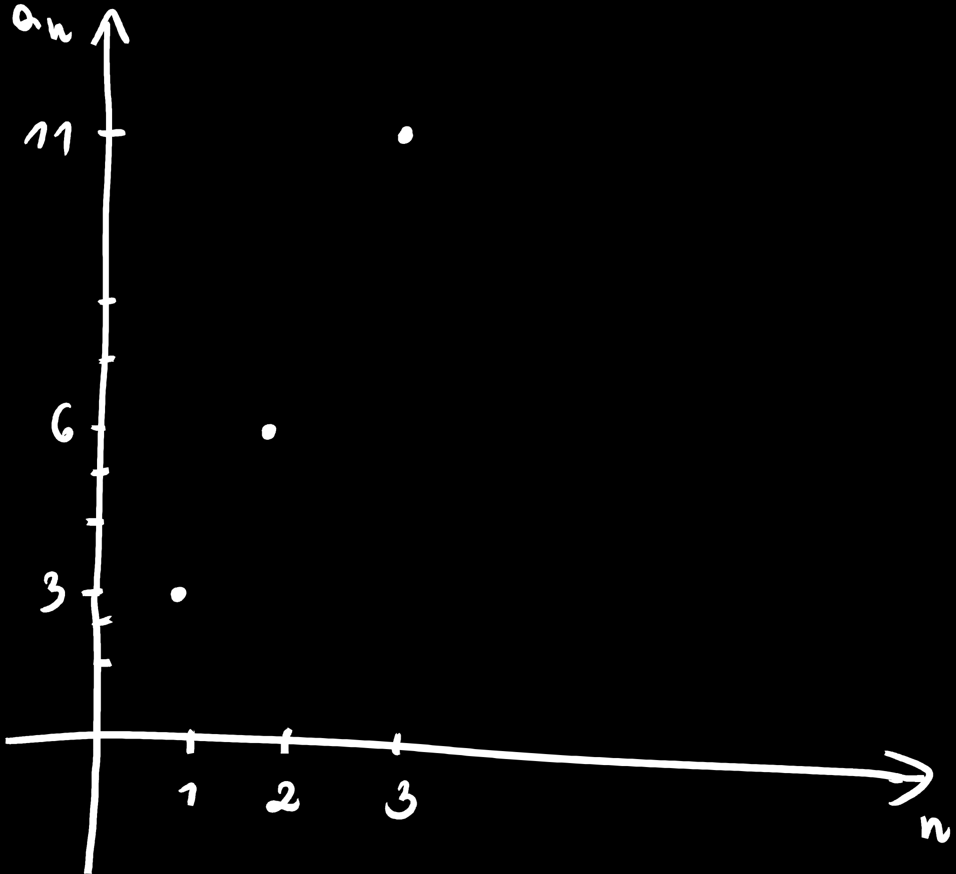
$1 < \frac{1}{50} n \quad / \cdot 50$

$50 < n \qquad n > 50$



GRANICE NIEWŁAŚCIWE

Ciąg (an) jest rozbieżny do ∞, jeśli dla każdej liczby M istnieje liczba naturalna k taka, że dla wszystkich n>k zachodzi nierówność an >M .



$a_n = n^2 + 2$

$a_1 = 3 \quad a_2 = 6 \quad a_3 = 11$

$a_3 = 3^2 + 2$

$g \in \mathbb{R}$

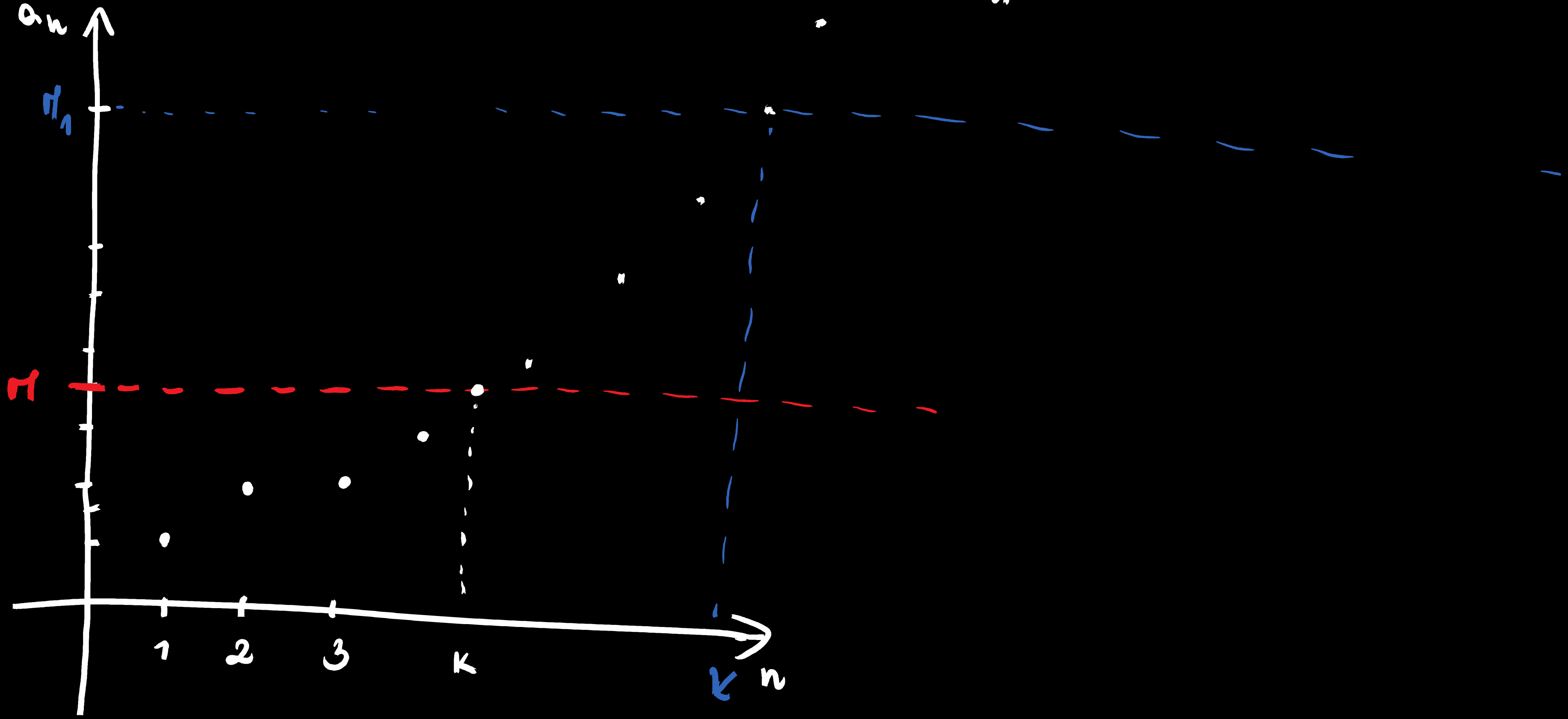
$\lim_{n \to \infty} (n^2 + 2) = \infty$

$\lim_{n \to \infty} (-n^2 + 2) = -\infty$

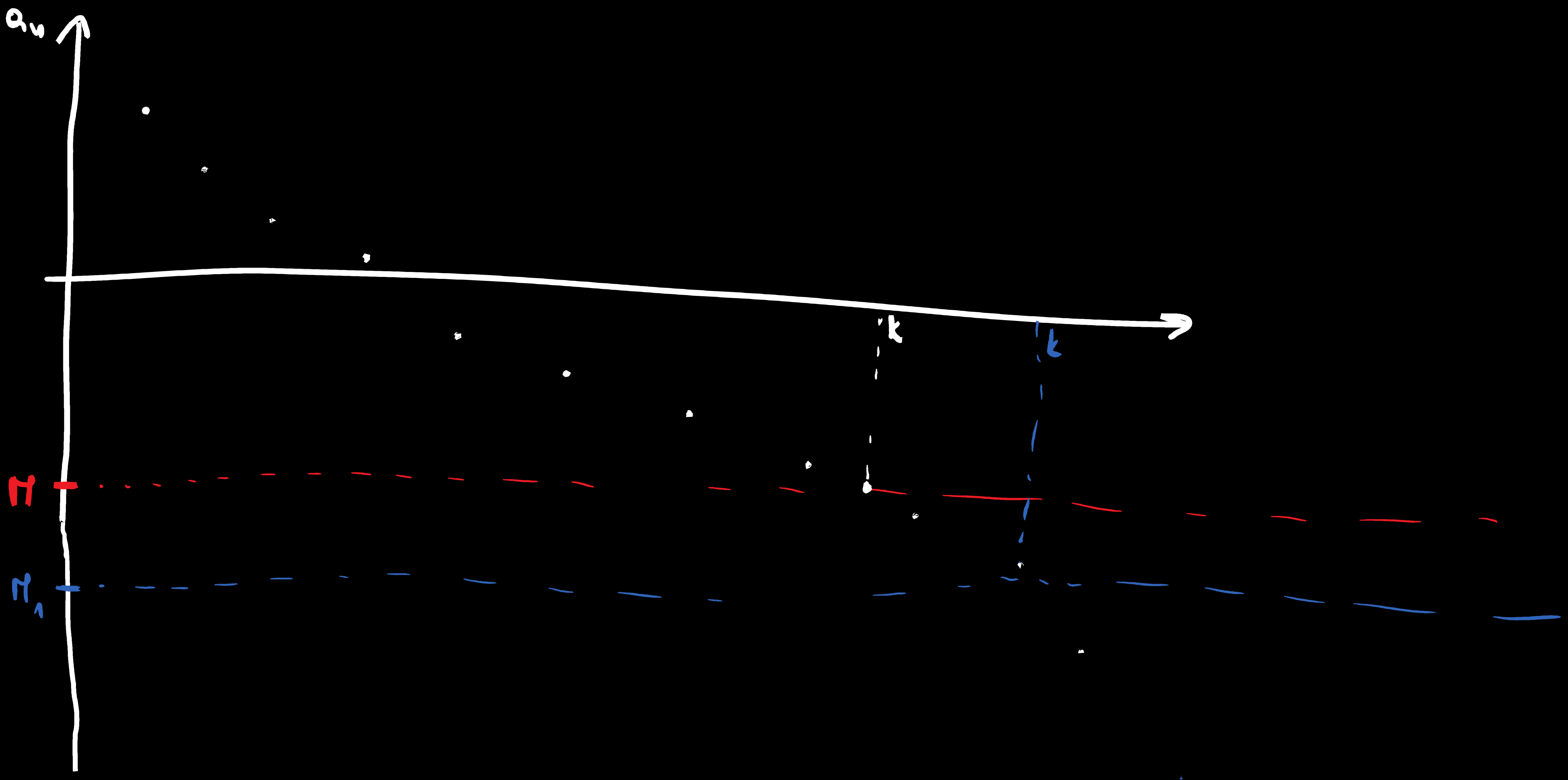
GRANICE NIEWŁAŚCIWE

Ciąg (an) jest rozbieżny do ∞, jeśli dla każdej liczby M istnieje liczba naturalna k taka, że dla wszystkich n>k zachodzi nierówność an >M .

$a_n > M$



Ciąg (an) jest rozbieżny do −∞, jeśli dla każdej liczby M istnieje liczba naturalna k taka, że dla wszystkich n>k zachodzi nierówność an <M



2b $a_n \in (M, \infty)$?

$n \in \mathbb{N}$

$a_n > M$

$a_{201} \ldots \in (20000, \infty)$

$$a_n = \frac{1}{2}n^2 \,, \quad M = 20\,000$$

$$a_n > M$$

$$\frac{1}{2}n^2 > 20\,000 \quad / \cdot 2$$

$$n^2 > 40\,000$$

$$n^2 - 40000 > 0$$

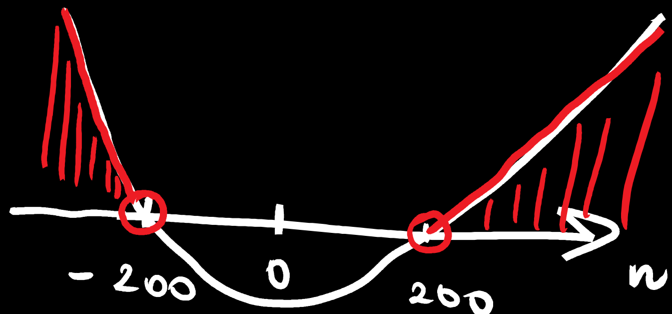
$$a^2 - b^2 = (a-b)(a+b)$$

$$(n - 200)\cdot(n + 200) > 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$n - 200 = 0$ $\qquad$ $n + 200 = 0$

$n = 200$ $\qquad$ $n = -200$



$$n \in \left[(-\infty, -200) \cup (200, \infty)\right] \cap \mathbb{N}$$

$$n \in \{201, 202, \ldots\}$$

1) $q > 1$, to $\lim\limits_{n \to \infty} q^n = \infty$

2) $k > 0$, to $\lim\limits_{n \to \infty} n^k = \infty$

3) $q \in (-1, 1)$, to $\lim\limits_{n \to \infty} q^n = 0$

4) $k > 0$, to $\lim\limits_{n \to \infty} \frac{1}{n^k} = 0$

1. $a_n = n^2 - n^3$

$$\lim_{n\to\infty}\left(n^2 - n^3\right) = [\infty - \infty] = \lim_{n\to\infty} \underbrace{n^3}_{\to\infty}\Big(\underbrace{\tfrac{1}{n}}_{\to 0} - \underbrace{1}_{\to 1}\Big) = \infty\cdot(0-1) =$$

$$= \infty \cdot (-1) = -\infty$$

$$\frac{1}{n} = \frac{\cancel{n}\cdot\cancel{n}}{\cancel{n}\cdot\cancel{n}\cdot n} = \frac{1}{n}$$

$n^3 \cdot \sqcup = n^3 \qquad n^3\cdot n^2 = n^5$

$$n^3\left(\frac{n^2}{n^3} - \frac{n^3}{n^3}\right)$$

1e $\quad a_n = 4^n - 6 \cdot 2^n - 100$

$$\lim_{n\to\infty} \left(4^n - 6 \cdot 2^n - 100\right) = [\infty - \infty - 100] =$$

$$= \lim_{n\to\infty} 4^n \cdot \left(\frac{4^n}{4^n} - \frac{6 \cdot 2^n}{4^n} - \frac{100}{4^n} \right) =$$

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

$$= \lim_{n\to\infty} \underbrace{4^n}_{\infty} \left(\underbrace{1}_{1} - \underbrace{6}_{6} \cdot \underbrace{\left(\frac{1}{2}\right)^n}_{0} - \underbrace{\frac{100}{4^n}}_{0} \right) =$$

$$= \infty \left(1 - 6 \cdot 0 - 0\right) = \infty \cdot 1 = \infty$$

1 f) $a_n = 3^n + 4^n - 12^n$

$$\lim_{n \to \infty} \left(3^n + 4^n - 12^n\right) = [\infty + \infty - \infty] =$$

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

$$= \lim_{n \to \infty} 12^n \left(\frac{3^n}{12^n} + \frac{4^n}{12^n} - \frac{12^n}{12^n}\right) =$$

$$= \lim_{n \to \infty} \underbrace{12^n}_{\infty} \left[\underbrace{\left(\frac{1}{4}\right)^n}_{0} + \underbrace{\left(\frac{1}{3}\right)^n}_{0} - \underbrace{1}_{1}\right] = \infty\,[0 + 0 - 1] = \infty \cdot (-1) =$$

$$= -\infty$$

2. $\lim\limits_{n\to\infty} \frac{(n^3-1) : n^2}{(2-n^2) : n^2} = \lim\limits_{n\to\infty} \frac{\frac{n^3}{n^2}-\frac{1}{n^2}}{\frac{2}{n^2}-\frac{n^2}{n^2}} = \lim\limits_{n\to\infty} \frac{\overset{\infty}{n}-\overset{0}{\frac{1}{n^2}}}{\underset{0}{\frac{2}{n^2}}-\underset{1}{1}} =$

$\frac{10:2}{12:2} = \frac{5}{6}$

$\frac{n\cdot n\cdot n}{n\cdot n} = n$

$= \frac{\infty - 0}{0-1} = \frac{\infty}{-1} = -\infty$

3.

$$\lim_{n \to \infty} \frac{(4^n - 1) :2^n}{(2^n + 6) :2^n} = \lim_{n \to \infty} \frac{\frac{4^n}{2^n} - \frac{1}{2^n}}{\frac{2^n}{2^n} + \frac{6}{2^n}} =$$

$$= \lim_{n \to \infty} \frac{\underbrace{2^n}_{\to \infty} - \underbrace{\frac{1}{2^n}}_{\to 0}}{\underbrace{1}_{\to 1} + \underbrace{\frac{6}{2^n}}_{\to 0}} = \frac{\infty - 0}{1 + 0} = \frac{\infty}{1} = \infty$$

Tw (o trzech ciągach)

$$\sqrt{10} = \sqrt{2}\cdot\sqrt{5}$$

$$\lim_{n\to\infty} \sqrt[n]{a} = 1$$

$$\lim_{n\to\infty} \sqrt[n]{2^n+3^n+4^n} = 4$$

$$\sqrt[n]{\underline{4}^n} \leqslant \sqrt[n]{2^n+3^n+\underline{4}^n} \leqslant \sqrt[n]{4^n+4^n+4^n}$$

$$\sqrt[n]{4^n} = 4 \xrightarrow{n\to\infty} \boxed{4}$$

$$\sqrt[n]{2^n+3^n+4^n} \xrightarrow{n\to\infty} 4$$

$$\sqrt[n]{4^n+4^n+4^n} = \sqrt[n]{3\cdot 4^n} = \sqrt[n]{3}\cdot\sqrt[n]{4^n} = \sqrt[n]{3}\cdot 4 \xrightarrow{n\to\infty} 1\cdot 4 = \boxed{4}$$

$$\lim_{n\to\infty} \frac{\left(\overset{a}{\sqrt{4n-3}} - \overset{b}{\sqrt{2n+10}}\right)}{1} = [\infty - \infty] =$$

$$(a-b)\cdot\underbrace{(a+b)} = a^2-b^2$$

$$= \lim_{n\to\infty} \frac{\left(\overset{a}{\sqrt{4n-3}} \overset{-}{-} \overset{b}{\sqrt{2n+10}}\right)\left(\overset{a}{\sqrt{4n-3}} \overset{+}{+} \overset{b}{\sqrt{2n+10}}\right)}{\sqrt{4n-3}+\sqrt{2n+10}} =$$

$$= \lim_{n\to\infty} \frac{4n-3-(2n+10)}{\sqrt{4n-3}+\sqrt{2n+10}} = \lim_{n\to\infty} \frac{4n-3-2n-10}{\sqrt{4n-3}+\sqrt{2n+10}} =$$

$$= \lim_{n\to\infty} \frac{(2n-13)\cdot\sqrt{n}}{\left(\sqrt{4n-3}+\sqrt{2n+10}\right)\cdot\sqrt{n}} = \left[\frac{\infty}{\infty}\right] = \lim_{n\to\infty} \frac{\frac{2n^{=\sqrt{n}\cdot\cancel{\sqrt{n}}}}{\cancel{\sqrt{n}}} - \frac{13}{\sqrt{n}}}{\sqrt{\frac{4n}{n}-\frac{3}{n}}+\sqrt{\frac{2n}{n}+\frac{10}{n}}} =$$

$$= \lim_{n\to\infty} \frac{\overset{\infty\leftarrow}{2\sqrt{n}} - \overset{\to 0}{\frac{13}{\sqrt{n}}}}{\sqrt{\underset{4\leftarrow}{4}-\underset{\to 0}{\frac{3}{n}}}+\sqrt{\underset{2\leftarrow}{2}+\underset{\to 0}{\frac{10}{n}}}} = \frac{\infty - 0}{2+\sqrt{2}} = \frac{\infty}{2+\sqrt{2}} = \infty$$